Modelling of convection in forest fires

M.I. Asensio, L. Ferragut
Dpto. Matemática Aplicada, Universidad de Salamanca

J. Simon
CNRS, Laboratoire de Mathématiques Appliquées, Université Blaise Pascal, Aubière cedex, France

November 2002
Convection Model in Forest Fires

We present a model coupling the fire propagation equations in a bidimensional domain representing the surface, and the air movement equations in a three dimensional domain representing an air layer.

As the air layer thickness is small compared with its length, an asymptotic analysis gives a three dimensional convective model governed by a bidimensional equation verified by a stream function.
Combustion Equations

Pyrolysis: Solid fuel, $y_s$

$$\partial_t y_s = -\beta_s y_s e^{\frac{\vartheta}{1+\epsilon_s \vartheta}}$$

Convection, Diffusion and Reaction: Gaseous fuel, $y_g$

$$\partial_t y_g + v \cdot \nabla_x y_g - \kappa \Delta_x y_g = -\beta y_g y_o e^{\frac{\vartheta}{1+\epsilon_g \vartheta}} + \beta_s y_s e^{\frac{\vartheta}{1+\epsilon_s \vartheta}} - \alpha g y_g$$

Convection, Diffusion and Reaction: Oxygen, $y_o$

$$\partial_t y_o + v \cdot \nabla_x y_o - \kappa \Delta_x y_o = -\beta y_g y_o e^{\frac{\vartheta}{1+\epsilon_g \vartheta}}$$

Convection, Diffusion and Reaction: Temperature, $\vartheta$

$$\partial_t \vartheta + v \cdot \nabla_x \vartheta - \nabla_x \cdot \left( \kappa_r \left( 1 + \epsilon_g \vartheta \right)^3 \nabla_x \vartheta \right) = y_g y_o e^{\frac{\vartheta}{1+\epsilon_g \vartheta}} - \alpha \vartheta$$

where radiation is modelled by a nonlinear diffusion term.
Wind model: Vertical diffusion model

\[ D = \{(x, z) : x \in d, \ h(x) < z < \delta \} \]
\[ S = \{(x, z) : x \in d, \ z = h(x)\} \]
\[ A = \{(x, z) : x \in d, \ z = \delta\} \]
\[ L = \{(x, z) : x \in \partial d, \ h(x) < z < \delta\} \]
The air velocity $\mathbf{U} = (U_1, U_2, U_3)$ and the potential $P$ satisfy the Navier–Stokes equations

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_{xz} \mathbf{U} - \frac{1}{\text{Re}} \Delta_{xz} \mathbf{U} + \nabla_{xz} P = \varphi Q \mathbf{e}_3$$

- The right-hand side represents the buoyancy forces.
- The density variations due to the temperature have been neglected into the other terms of the equation.
- The air compressibility is also neglected, $\nabla_{xz} \cdot \mathbf{U} = 0$
Derivation of the Wind model

- On surface $S$, $\mathbf{U} \cdot \mathbf{N} = 0$, $\left. \frac{\partial \mathbf{U}}{\partial N} \right|_{tang} = \zeta \mathbf{U}$
- On the air upper boundary $A$, $\mathbf{U} \cdot \mathbf{N} = 0$, $\left. \frac{\partial \mathbf{U}}{\partial N} \right|_{tang} = 0$
- On the air side boundary $L$, $\mathbf{U}|_L = (v_m, 0)$ where $\partial_z v_m = 0$, $\int_{\partial d}(\delta - h)v_m \cdot \mathbf{n} ds = 0$
The layer thickness is small in relation to its width $\delta \ll 1$.

The wind is not too strong $\delta^2 \text{Re} \ll 1$.

Preserving only the dominant terms and rescaling $P$, being $\delta = \varphi \text{Re}$

<table>
<thead>
<tr>
<th>Equations</th>
<th>Boundary conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\partial^2_{zz} \mathbf{V} + \nabla_x P = 0$</td>
<td>$\partial_z \mathbf{V} = \zeta \mathbf{V}$, $(\mathbf{V}, W) \cdot \mathbf{N} = 0$ on $S$</td>
</tr>
<tr>
<td>$\partial_z P = \varphi Q$</td>
<td>$\partial_z \mathbf{V} = 0$, $W = 0$ on $A$</td>
</tr>
<tr>
<td>$\nabla_x \cdot \mathbf{V} + \partial_z W = 0$</td>
<td>$\mathbf{V} \cdot \mathbf{n} = v_m \cdot \mathbf{n}$ on $\partial d$</td>
</tr>
</tbody>
</table>
Derivation of the Wind model

- Defining the horizontal flux at a point \( \mathbf{x} \in d \) and time \( t \) by
  \[
  \overline{\mathbf{V}}(t, \mathbf{x}) = \int_{h(x)}^{\delta} \mathbf{V}(t, \mathbf{x}, z) \, dz
  \]
- The incompressibility of \( \mathbf{U} \) and the fact that the air does not cross \( S \) and \( A \) give the incompressibility of the horizontal flux
  \[
  \nabla_x \cdot \overline{\mathbf{V}} = 0
  \]
- By Stokes \( \int_{\partial d} \overline{\mathbf{V}} \cdot \mathbf{n} \, ds = \int_d \nabla_x \cdot \overline{\mathbf{V}} \, dx = 0 \) so as
  \[
  \overline{\mathbf{V}} = (\delta - h)\mathbf{v}_m \text{ on } \partial d \text{ we need the hypothesis}
  \]
  \[
  \int_{\partial d} (\delta - h)\mathbf{v}_m \cdot \mathbf{n} \, ds = 0
  \]
The temperature $q$ considered in the combustion equations is obviously the value of $Q$ on the surface, that is
\[ q(t, x) = Q(t, x, h(x)) \]
We assume that
\[ Q(t, x, z) = q(t, x) \frac{\delta-z}{\delta-h(x)} \]
The velocity $v$ considered in the combustion equations is the value of $V$ on the surface, that is, the horizontal component of the wind velocity $U$ on the surface
\[ v(t, x) = V(t, x, h(x)) \]
Derivation of the Wind model

Compute explicitly $P(t, x, z)$ and $V(t, x, z)$ in terms of a 2D potential $p$.

For a fixed $x$, equation $\partial_z P = \lambda Q$ provides

$$P(t, x, z) = p(t, x) + \frac{\lambda q(t, x)}{\delta - h(x)} \left( \delta z - \frac{1}{2} z^2 \right)$$

Equation $\partial_{zz}^2 V = \nabla_x P$ together with the boundary conditions, provides

$$V(x, z) = m(x, z) \nabla p(x) + n(x, z) \nabla T(x)$$
Vertical diffusion model

Gives the horizontal wind field in a 3D domain by the expression

\[ V(x, z) = m(x, z) \nabla p(x) + n(x, z) \nabla \hat{T}(x) \]

1. \[ m(x, z) = \frac{1}{2} z^2 - \delta z - \frac{1}{2} h^2(x) + (\delta + \xi) h(x) - \xi \delta \]
2. \[ n(x, z) = -\frac{1}{24} z^4 + \frac{1}{6} \delta z^3 - \frac{1}{3} \delta^3 z + \frac{1}{24} h^4(x) - \ldots \]
3. \( p(x) \) is a potential function
The potential $p(x)$ satisfies the following boundary problem

\[-\nabla(a \nabla p) = \nabla(b \nabla \hat{T}) \quad \text{in } d\]

\[a \frac{\partial p}{\partial n} = -b \frac{\partial \hat{T}}{\partial \nu} + (\delta - h)v_m.\nu \quad \text{on } \partial d\]

1. $a(x) = \frac{1}{3}(\delta - h(x))^2(3\xi + \delta - h(x))$
2. $b(x) = \frac{1}{30}(\delta - h(x))^2 \left(2\delta^2(2\delta + 5\xi) - 2\delta(\delta - 5\xi)h(x) - (3\delta + 5\xi)h^2(x) + h^3(x)\right)$
Optimal control problem

Let \( v = (\delta - h)v_m \nu \) the flow on the boundary

We formulate the former problem as an optimal control problem

Given \( N \) experimental measurements of the wind velocity \( V_i, i = 1, \ldots, N \), at \( N \) given points \( P_i = (x_i, z_i), i = 1, \ldots N \), we search for the value of \( v \in L^2_0(\partial d) \) such that the value \( V(x_i, z_i) \) given by the expression \( V(x, z) = m(x, z)\nabla p(x) + n(x, z)\nabla \hat{T}(x) \) are as close as possible to the experimental values \( V_i \).
Optimal control problem

1. $v \in L^2_0(\partial \omega)$ is the control.

2. 

\[-\nabla(a\nabla p) = \nabla(b\nabla \hat{T}) \text{ in } \omega\]

\[a \frac{\partial p}{\partial n} = -b \frac{\partial \hat{T}}{\partial \nu} + (\delta - h)v_m \nu \text{ on } \partial d\]

are the state equations.

3. 

\[J(v) = \frac{1}{2} \sum_i \int_\omega \rho_{\epsilon,(x-x_i)}(x) \left( m(x, z)\nabla p(x) + n(x, z_i)\nabla q(x) - V_i \right)^2 + \frac{\alpha}{2} \int_{\partial d} v^2 \text{ is the cost function}\]
The optimal control problem is characterised by

\[
\int_{\omega} a \nabla p(u) \nabla \varphi + \frac{1}{\alpha} \int_{\partial \omega} q \varphi = - \int_{\omega} b \nabla \hat{T} \nabla \varphi \quad \forall \varphi \in V
\]

\[
\int_{\omega} a \nabla q(u) \nabla \psi \\
- \sum_{i=1}^{N} \int_{\omega} \rho \epsilon (x - x_i)(m \nabla p(u) + n \nabla \hat{T} - V_i)m \nabla \psi = 0 \quad \forall \psi \in V
\]

\[
u = -\frac{1}{\alpha} q \quad \text{on} \quad \partial d
\]
4.3. Calculus of wind

Practically, we calculate the unique solution \((p, q)\) of (approached) coupled equations (42)–(43) for a small parameter \(\eta\), and then we calculate the (approached adjusted) wind velocity \(V(x, z)\) at every point \((x, z)\) in terms of \(\nabla p(x)\) using expression (16).

4.4. Finite element approximation

Let us discretize the approached equations (42)–(43). Let \(T_H\) be a uniform triangulation of \(\omega\) corresponding to a discretization parameter \(H\) and let \(V_H\) be the associated space of \(P_1\) (or \(P_2\)) finite elements. Besides a better order of convergence, a reason in favor of \(P_2\) against \(P_1\) is that in practical applications, the variable of physical interest is the wind velocity \(V\) which is obtained from the potential \(p\) using expression (16), involving derivatives.

Choosing a finite element basis \(\{\phi_i\}\) for \(V_H\), we introduce the following matrices

\[
G = \left\{ \int_\omega a \nabla \phi_r \cdot \nabla \phi_k + \eta \int_{\partial \omega} \phi_r \phi_k \right\}_{r,k},
\]

\[
C_1 = \left\{ \frac{1}{\alpha} \int_{\partial \omega} \phi_r \phi_k \right\}_{r,k},
\]

\[
C_2 = \left\{ \sum_{i=1}^N \int_\omega \rho_{r,i} m^2 \nabla \phi_r \cdot \nabla \phi_k \right\}_{r,k}
\]

and the vectors

\[
f_p = \left\{ - \int_\omega b \nabla t \cdot \nabla \phi_r \right\}_r,
\]

\[
f_q = \left\{ - \sum_{i=1}^N \int_\omega \rho_{r,i} (n \nabla t - V_i) m \cdot \nabla \phi_r \right\}_r.
\]

Then, the discrete problem associated to (42)–(43) is the following linear algebraic system:

\[
\begin{bmatrix}
G & C_1 \\
-C_2 & G
\end{bmatrix}
\begin{bmatrix}
p \\
q
\end{bmatrix}
= \begin{bmatrix}
f_p \\
f_q
\end{bmatrix}
\tag{49}
\]

The matrix in (49) being nonsymmetric and very ill-conditioned, most of the standard iterative methods fail to converge or have a very slow convergence (this is the case of GMRES-ILU preconditioned). For moderate number of unknowns we use the state-of-the-art sparse
LU factorization [4]. In [5] a highly effective solution method is obtained by means of a preconditioned Schur complement approach, leading to a nonsymmetric system that can be solved by GMRES in a constant number of iterations. For the description and a complete numerical analysis of this approach see [5].

5. NUMERICAL EXAMPLES

5.1. Example 1: Effect of a topography and of a temperature gradient

In this section we consider the effect of two hills on the wind, as well as the effect of the temperature gradient in a square of 6 by 6 kilometers. The ground height and ground temperature are shown on Figure 1.

![Surface Contours](IsoValue)

![Temperature Contours](IsoValue)

Figure 1. Ground height and ground temperature (Example 1)

The wind velocity is supposed to be known (by experimental measurements) at four points of horizontal coordinates \(x = (1, 1), (5, 1), (5, 5), (1, 5)\) and of height \(z = 0.1 + h(x)\), with
the same value $V(x, z) = (2, 0)$ and we take $\alpha = 0.001$.

Figure 2 shows the calculated adjoint state and potential, and Figure 3 shows the calculated velocity module and wind field on the ground surface, that is for $z = h(x)$. As expected, the

Figure 2. Adjoint state and potential (Example 1)

Figure 3. Velocity module and wind field (Example 1)
Figure: Surface

Figure: Position of control points
Functional relationship between GIS data and model parameters

Relationship between rugosity and the friction coefficient of in the wind model

The inverse $\xi$ of the friction coefficient as a function of the rugosity $\rho$ is given by

$$\xi = 0.25(1 + 0.05\rho - 0.01\rho^2)$$
Figure: E208-15m

Figure: E212-30m
Figure: E242-40m

Figure: E283-49m
Figure: E212-15m-dir.pdf

Figure: E212-30m-dir.pdf
Domain, contours of the surface and meteorologic wind

Surface function

\[ h(x_1, x_2) = 0.5e^{(-100(x-1)^2 - 50(y-0.5)^2)} \]
Numerical simulation

Initial Temperature

Initial fire focus

\[ \vartheta(x_1, x_2) = 30e^{-200(x_1-0.25)^2 + (x_2-0.5)^2} \]

Parameters: \( \varepsilon_s = 0.05, \varepsilon_g = 0.04, \beta_s = 1, \beta = 1/1.2, \kappa = 0.1, \kappa_r = 0.1 \)
\( \lambda = 3, \xi = 0.1 \)
Numerical simulation

Contours of the Temperature after 1000 time steps
Numerical simulation

Contours of the concentration of Oxygen after 1000 time steps
Numerical simulation

Contours of the concentration of gaseous fuel after 1000 time steps
Numerical simulation

Contours of the concentration of solid fuel after 1000 time steps
Numerical simulation

Contour plot of the $x$-component of velocity after 600 time steps
Numerical simulation

Contour plot of the $x$-component of velocity velocity after 1000 time steps
Numerical simulation

Contour plot of the $y$-component of velocity after 1000 time steps
Numerical simulation

Finite Element Mesh after 1000 time steps